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THE CONCEPT OF MULTISET

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1. Introduction

Multisets have been used in mathematics ([4, p. 636]), computer science ([1]), and logic ([6]). Definitions for various concepts involving multisets have been given by KNUTH [4, p. 464] and HICKMAN [2]; however these definitions tend to be given in a rather *ad hoc* fashion, and indeed HICKMAN states that he has chosen his definitions to fit results already used by those who have worked with multisets. There is of course nothing wrong with this, but if the concept of multiset is a reasonable one there should be some underlying idea which leads naturally to the various definitions. It is the contention of this paper that the intuitive concept of multiset in fact contains *two* underlying ideas, and that these ideas should be separated. One of the resulting concepts is more set-like than the other, and the name "multiset" has been appropriated for this concept; the other concept is more numeric in character and has been named "multinumber". It is hoped that distinguishing the concepts "multiset" and "multinumber" will clear up some unsatisfactory features in previous treatments of multisets.

In this paper most attention is paid to the concept "multiset", as it appears to be more fundamental. The multisets (in the sense of this paper) form a category ***Mul*** in a natural way, and this fact is used to generate definitions for multiset concepts by applying category-theoretic definitions to ***Mul***. This is done in Section 2. In Section 3 multinumbers are discussed and compared with multisets. Finally Section 4 contains some further remarks about the category ***Mul***.

Most of the category-theoretic terminology used is taken from MACLANE's book [5].

2. Multisets

Intuitively a multiset is like a set, but may have repeated occurrences of elements. Thus $[a, b]$ and $[a, a, b]$ are distinct multisets (where square brackets are used to indicate that what is being considered is a multiset, not an ordinary set like $\{a, b\}$). We introduce a change of view, and regard $[a, a, b]$ as being really of the form $[a, a', b]$, where a and a' are different objects of the same *sort*, whereas b is of a different sort from a and a'. This new viewpoint may perhaps be seen as not doing justice to the intuitive idea of multiset; another approach is discussed in Section 3. The viewpoint of the present section leads to the following formal definitions.

Definition 2.1. A *multiset* X is a pair $\langle X_0, \varrho\rangle$, where X_0 is a set and ϱ an equivalence relation on X_0. The set X_0 is called the *field* of the multiset. Elements of X_0

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in the same equivalence class will be said to be *of the same sort*; elements in different equivalence classes will be said to be *of different sorts*.

Definition 2.2. Let $X = \langle X_0, \varrho\rangle$ and $Y = \langle Y_0, \sigma\rangle$ be multisets. A *morphism* of multisets is a function $f\colon X_0 \to Y_0$ which *respects sorts*; that is if $x, x' \in X_0$ and $x\varrho x'$, then $f(x)\,\sigma f(x')$.

We generally write $f\colon X \to Y$ for a multiset morphism, suppressing explicit mention of X_0 and Y_0. (If a multiset X is mentioned, its field will always be called X_0.) The category of multisets and multiset morphisms will be denoted ***Mul***. In this section we consider what some basic definitions of category theory come to in the category ***Mul***.

Proposition 2.3. *Let $X = \langle X_0, \varrho\rangle$ and $Y = \langle Y_0, \sigma\rangle$ be multisets. Let $f\colon X \to Y$ be a multiset morphism; we write $f\colon X_0 \to Y_0$ for the function between the fields. Then*

(i) *$f\colon X \to Y$ is a monomorphism in **Mul** iff $f\colon X_0 \to Y_0$ is one-to-one;*

(ii) *$f\colon X \to Y$ is an epimorphism in **Mul** iff $f\colon X_0 \to Y_0$ is onto;*

(iii) *$f\colon X \to Y$ is an isomorphism in **Mul** iff $f\colon X_0 \to Y_0$ is a bijection and also has the property that $x\varrho x'$ iff $f(x)\,\sigma f(x')$.* □

As an example consider $f\colon [a, b] \to [c, c']$, where a and b are of different sorts, c and c' are of the same sort, $f(a) = c$ and $f(b) = c'$. Then f is a multiset morphism, and is both a monomorphism and an epimorphism, but not an isomorphism. (When elements of multisets are considered, elements of distinct sorts will generally be denoted by distinct letters, and elements of the same sort will be denoted by the same letter, dashes distinguishing different elements of that sort.)

Hickman [2] considers the Schröder-Bernstein theorem in the case of multisets. For ordinary sets the Schröder-Bernstein theorem asserts that if $f\colon X \to Y$ and $g\colon Y \to X$ are both one-to-one, then there is a bijection between X and Y. The corresponding result fails for multisets in general, and we may use an example similar to Hickman's to show this (though Hickman's definitions of "multiset" and "multiset morphism" differ from the ones used here).

Example 2.4. Let X be the multiset $[x_1, x_2, x_2', x_3, x_3', x_3'', \ldots]$ (with n "copies" of x_n, and x_m, x_n being of different sorts if $m \neq n$) and Y the multiset $[y_1, y_1', y_2, y_2', y_2'', y_3, y_3', y_3'', y_3''', \ldots]$ (with $n + 1$ copies of y_n). Define $f\colon X \to Y$ by $f(x_n^{(i)}) = y_n^{(i)}$ and $g\colon Y \to X$ by $g(y_n^{(i)}) = x_{n+1}^{(i)}$. Then f and g are both monomorphisms, but clearly there is no isomorphism between X and Y.

It is also possible to give a counter-example to the Schröder-Bernstein theorem involving multisets with only a finite number of sorts, there being infinitely many elements of some sorts. The Schröder-Bernstein theorem does hold for *finite* multisets, that is multisets with finite fields.

Proposition 2.5.

(i) *The initial object of **Mul** is the empty set together with the empty equivalence relation.*

(ii) *The cartesian product $X \times Y$ of two multisets $X = \langle X_0, \varrho\rangle$ and $Y = \langle Y_0, \sigma\rangle$ has as field the product set $X_0 \times Y_0$; the equivalence relation relates $\langle x_1, y_1\rangle$ and $\langle x_2, y_2\rangle$ iff both $x_1 \varrho x_2$ and $y_1 \sigma y_2$.*

(iii) *The coproduct* $X \sqcup Y$ *of* $X = \langle X_0, \varrho \rangle$ *and* $Y = \langle Y_0, \sigma \rangle$ *has as field* $X_0 \mathbin{\dot\cup} Y_0$ *(the disjoint union of* X_0 *and* Y_0*) and equivalence relation* τ*, where: if* $a, b \in X_0 \mathbin{\dot\cup} Y_0$*, then* $a\tau b$ *iff either* $a, b \in X_0$ *and* $a\varrho b$ *or* $a, b \in Y_0$ *and* $a\sigma b$. □

It seems reasonable to call the initial object of ***Mul*** the *empty multiset*, and to denote it by $\emptyset$. It also seems reasonable to call $X \sqcup Y$ the *disjoint union* of X and Y.

Subobjects. The ordinary notion of subobject appears not to be very useful: for example $m\colon [a, b] \to [a, a']$ where $m(a) = a$, $m(b) = a'$ is a monomorphism and so determines a subobject. We get a better notion by considering only certain monomorphisms.

A *strong monomorphism* $m\colon A \to B$ is a monomorphism such that if $e\colon C \to D$ is an epimorphism and $f\colon C \to A$, $g\colon D \to B$ are such that $ge = mf$, then there is $u\colon D \to A$ such that $ue = f$ and $mu = g$. Diagrammatically,

$$\begin{tikzcd} C \arrow[r, "e"] \arrow[d, "f"'] & D \arrow[d, "g"] \arrow[dl, dashed, "u"'] \\ A \arrow[r, "m"'] & B \end{tikzcd}$$

In ***Mul*** a strong monomorphism $f\colon X \to Y$ is a monomorphism such that if $a, b \in X$ and $f(a)$ and $f(b)$ are of the same sort, then a and b are of the same sort. We define a *strong subobject* to be an equivalence class of strong monomorphisms, the equivalence being defined in the same way as for ordinary subobjects.

Definition 2.6. Let $X = \langle X_0, \varrho \rangle$ be a multiset. A *submultiset* $Y \subseteqq X$ is a multiset of the form $\langle Y_0, \sigma \rangle$ where Y_0 is a subset of X_0 and σ is ϱ restricted to Y_0.

It is not hard to see that in ***Mul*** every strong subobject of X is determined by a unique submultiset (in the sense above) of X. So the notion of strong subobject gives a satisfactory notion of "submultiset". We also note that the empty multiset $\emptyset$ is the smallest strong subobject of every multiset.

Let Z be an object in an arbitrary category. If X and Y are strong subobjects of Z the *strong union* of X and Y is defined to be the least strong subobject of Z greater than or equal to both X and Y (in the partial ordering on strong subobjects), provided such a least strong subobject exists. The ordinary intersection of strong subobjects is always strong.

Proposition 2.7. *Let* X *and* Y *be submultisets of* Z*. Define* $X \cup Y$ *and* $X \cap Y$ *to be the submultisets of* Z *with fields* $X_0 \cup Y_0$ *and* $X_0 \cap Y_0$*, respectively. Then* $X \cup Y$ *is the strong union of* X *and* Y*, and* $X \cap Y$ *is the intersection of* X *and* Y. □

There seems to be no point in trying to form the union of multisets which are not submultisets of the same multiset: for example, what would be the union of $\langle \{a, b\}, \varrho \rangle$ and $\langle \{a, b\}, \sigma \rangle$, where $a\varrho b$ but it is not the case that $a\varrho b$?

Complementation of multisets presents no difficulties. Let X be a submultiset of Z; let $Z - X$ be the submultiset of Z with field $Z_0 - X_0$.

Proposition 2.8. *The strong union of* X *and* $Z - X$ *is* Z*; the intersection of* X *and* $Z - X$ *is* $\emptyset$. □

Let Z be a multiset. The collection of submultisets of Z is in bijection with the collection of (ordinary) subsets of the field Z_0 of Z, and the behaviour of the submultisets of Z under the operations $\cup$, $\cap$ and $-$ of Propositions 2.7 and 2.8 is exactly the same as that of the subsets of Z_0 under the corresponding operations for subsets. Thus the operations $\cup$, $\cap$ and $-$ for submultisets enjoy all of the properties of the corresponding operations for subsets. In particular the collection of all the submultisets of Z forms a complete Boolean algebra.

Exponentiation. The category ***Mul*** is cartesian closed: if $X = \langle X_0, \varrho \rangle$ and $Y = \langle Y_0, \sigma \rangle$ are multisets, then Y^X may be taken to be the multiset with field the set of all multiset morphisms from X to Y and equivalence relation τ, where $f \tau g$ iff $x_1 \varrho x_2$ implies $f(x_1)\, \sigma g(x_2)$ for all $x_1, x_2 \in X_0$.

Mul also has a strong subobject classifier Ω, which may be taken to be any multiset of the form $[a, a']$ (with a and a' of the same sort). Given any multiset X, we may form Ω^X, which should play the rôle of a "power multiset".

Proposition 2.9. *Let X be a multiset. Define $\mathscr{P}X$ (the power multiset of X) to be the multiset with field all the submultisets of X, and with equivalence relation such that every pair of elements in the field are related (i.e. $\mathscr{P}X$ has only one sort). Then $\mathscr{P}X \cong \Omega^X$.*

An analogue of Cantor's theorem now holds for multisets: there is a monomorphism $X \to \mathscr{P}X$, but no monomorphism $\mathscr{P}X \to X$. The definition of $\mathscr{P}X$ seems rather counter-intuitive, and is the first concept we have encountered for multisets which is not a generalization of the corresponding concept for ordinary sets. Nonetheless the definition of $\mathscr{P}X$ arises naturally from those of Ω and "submultiset".

The definition of "submultiset" in Definition 2.6 is not the only possible generalization of the concept of subset to ***Mul***. Another one is the following.

Definition 2.10. Let Z be a multiset. A *replete submultiset* X of Z is a submultiset X of Z such that if $z \in X$ and $z' \in Z$ and z, z' are of the same sort, then $z' \in X$. That is, if X contains one element of a given sort, then it must contain all the elements of that sort in Z.

Let us now define 2 to be the multiset $[0, 1]$, where 0 and 1 are of different sorts. Then 2 is a "replete submultiset classifier". We may define the "replete power multiset" $\mathscr{R}X$ to be the multiset with field all the replete submultisets of X, and with equivalence relation such that every element of $\mathscr{R}X$ is of a different sort. Then $\mathscr{R}X$ is isomorphic to 2^X, and directly generalizes the ordinary notion of power set. However "submultiset" seems to be a better notion than "replete submultiset".

Another definition involving multisets is a form of "disjoint union" differing from that in Proposition 2.5.

Definition 2.11. Let X and Y be submultisets of $Z = \langle Z_0, \varrho \rangle$. $X \overset{+}{\cup} Y$ is the submultiset with field $X_0 \times \{0\} \cup Y_0 \times \{1\}$ and equivalence relation τ, where

$$\begin{aligned} &\text{if } x_1, x_2 \in X_0 \text{ then } \langle x_1, 0\rangle \tau \langle x_2, 0\rangle \text{ iff } x_1 \varrho x_2, \\ &\text{if } y_1, y_2 \in Y_0 \text{ then } \langle y_1, 1\rangle \tau \langle y_2, 1\rangle \text{ iff } y_1 \varrho y_2, \\ &\text{if } x \in X_0 \text{ and } y \in Y_0 \text{ then } \langle x, 0\rangle \tau \langle y, 1\rangle \text{ iff } x \varrho y \\ &\qquad\qquad\qquad\qquad\text{and } \langle y, 1\rangle \tau \langle x, 0\rangle \text{ iff } x \varrho y. \end{aligned}$$

Note that $X \overset{+}{\cup} Y$ need not be a submultiset of Z. For example, if $Z = [a, b, c]$, $X = [a, b]$ and $Y = [b, c]$, then

$$X \overset{+}{\cup} Y = [\langle a, 0\rangle, \langle b, 0\rangle, \langle b, 1\rangle, \langle c, 1\rangle]$$

which is of the form $[a, b, b', c]$. The notation $X \overset{+}{\cup} Y$ is KNUTH's ([4, p. 454]).

All the concepts in this section except those of Definitions 2.10 and 2.11 arise naturally out of general category theory and make good sense for multisets. This suggests that the category ***Mul*** is a reasonable explication of the notion of multiset. The concepts of Definitions 2.10 and 2.11 can be treated category-theoretically: further discussion is deferred until Section 4.

3. Multinumbers

A view of multisets which differs from that in Section 2 is that a multiset like $[a, a, b]$ is really a set where the elements are labelled with numbers, thus: $\{a^2, b^1\}$. (We may regard a and b as sorts in the sense of Section 2.) This view is arguably closer to the intuitive conception of multiset. In this section we restrict ourselves to multisets in which each element occurs only a finite number of times. This is not an essential restriction, and can be removed in what follows by replacing N by the collection of all cardinal numbers (including infinite cardinals). The view that a multiset is a set with the elements labelled by numbers can be formulated in several more or less equivalent ways, one of which is the following. Let $\mathscr{S}$ be the collection of all the sorts which can occur in multisets.

Definition 3.1. A *multinumber* is a function from $\mathscr{S}$ to N.

To set theorists this definition may have an air of illegitimacy about it; we discuss this point at the end of the section. If f is a multinumber, $a \in \mathscr{S}$ and $f(a) = 0$, this is interpreted as saying that a does not occur in f. Clearly f could be replaced by a smaller object carrying the same information, for example f restricted to the set $\{a \in \mathscr{S}\colon f(a) > 0\}$. However Definition 3.1 is convenient notationally.

The collection of all multinumbers may be written $\mathsf{N}^{\mathscr{S}}$. N has operations $+$ and $\cdot$ defined on it, and has an ordering consistent with these operations. Furthermore the ordering is a lattice order (with no greatest element), the lattice operations being max and min. All this can be expressed by saying that N is a lattice-ordered semiring (where the "semi" part means that additive inverses do not exist). $\mathsf{N}^{\mathscr{S}}$ is also a lattice-ordered semiring, the concepts in $\mathsf{N}^{\mathscr{S}}$ being defined coordinate-wise, as follows.

Definition 3.2. Let f and g be multinumbers.

(i) We say $f \leqq g$ if $f(a) \leqq g(a)$ for all $a \in \mathscr{S}$.

(ii) $f + g$ is the multinumber given by $(f + g)(a) = f(a) + g(a)$ for all $a \in \mathscr{S}$.

(iii) $f \cdot g$ is given by $(f \cdot g)(a) = f(a) \cdot g(a)$.

(iv) $\max(f, g)$ is given by $(\max(f, g))(a) = \max(f(a), g(a))$.

(v) $\min(f, g)$ is given by $(\min(f, g))(a) = \min(f(a), g(a))$.

(vi) If $f \leqq g$ as in (i), define $g - f$ by $(g - f)(a) = g(a) - f(a)$ for all $a \in \mathscr{S}$.

The ordering given in (i) above is of course only a partial ordering. Relationships valid for N, such as the distributive law, will continue to be valid for the concepts of Definition 3.2.

Since $\mathsf{N}^{\mathscr{S}}$ inherits much of its structure from N it seems reasonable to call the elements of N *multinumbers*. There does not seem to be any particular point in trying to view $\mathsf{N}^{\mathscr{S}}$ as a category; rather it should be considered as a partially ordered algebra. The elements of $\mathsf{N}^{\mathscr{S}}$ should thus behave more like numbers than sets.

A multiset (in the sense of Section 2) has an *associated multinumber* defined in the evident way: for example $[a, a', b]$ has associated multinumber f, where $f(a) = 2$, $f(b) = 1$ and $f(c) = 0$ for all c in $\mathscr{S}$ distinct from a and b. We write $X^{\#}$ for the multinumber associated with the multiset X. Some of the concepts in Definition 3.2 can be regarded as being induced by operations on multisets. Thus, if X is a submultiset of Y, then $X^{\#} \leq Y^{\#}$ in the sense of Definition 3.2 (i), and the operation $X \overset{+}{\cup} Y$ of Definition 2.11 induces the addition operation of Definition 3.2 (ii). The product $X \times Y$ of Proposition 2.5 does induce an operation on multinumbers, but it is not a coordinate-wise operation and so is quite different from the operation of Definition 3.2 (iii). However some operations on multisets do not induce operations on multinumbers. This is true in particular of the operations of union, intersection and complement of Propositions 2.7 and 2.8. Conversely, some operations on multinumbers are not induced by any operations on multisets. This is true of the max, min and subtraction operations of Definition 3.2 parts (iv), (v) and (vi).

Some of the difficulties with the theory of multisets seem to come from the use of a rather indiscriminate mixture of what are here distinguished as multiset concepts and multinumber concepts. Thus what Hickman [2] calls "multisets" are actually multinumbers in the terminology of the present paper, but Hickman calls the max, min and subtraction operations of Definition 3.2 "union", "intersection" and "complement" respectively, and compares these operations with the operations on sets which have the same names. However the operations of Definition 3.2 are more like operations on numbers than operations on sets. Another paper which uses multinumbers extensively (again under the name "multisets") is that of Meyer and McRobbie [6]. Meyer and McRobbie consider several concepts, mostly defined coordinate-wise, and including the concepts of Definition 3.2 parts (i), (ii) and (vi) (under different names). Although Meyer and McRobbie use the name "multiset" they are quite clear that the properties they use are basically arithmetical, and close their paper with a call to add the other natural numbers to the 0 and 1 of Boolean algebra, suggesting that excessive reliance on Boolean-algebraic properties has been somewhat crippling. From the point of view of the present paper, Boolean properties should apply to multisets (and do: see the discussion after Proposition 2.8), while properties like those of numbers should apply to multinumbers.

The discussion above may be summed up by the statement that the concept of multinumber is related to that of multiset in the same way that the concept of (cardinal) number is related to the ordinary concept of set. I think that the concept of multiset is more fundamental than that of multinumber, but the concept of multinumber may be the more useful of the two. In any event the concepts should be distinguished.

We close this section with a comment on the set-theoretic illegitimacy mentioned after Definition 3.1. In standard set theory the only elements of sets are again sets, so $\mathscr{S}$ is the collection of all sets. Then $\mathsf{N}^{\mathscr{S}}$ is too large for standard set theory to consider. One way out is to regard $\mathscr{S}$ as the set of *small* sets in an appropriate sense (for example the set of sets of rank less than some inaccessible cardinal). Another, possibly better, solution is to consider only those functions from $\mathscr{S}$ to N which take value zero everywhere exept on a sub*set* of $\mathscr{S}$. The content of the present section is not altered, whichever solution is adopted.

4. Further category-theoretic comments

The category ***Mul*** of Section 2 is by no means new, though its identification as the category of multisets appears to be. Indeed ***Mul*** is equivalent to the category whose objects are epimorphisms in ***Set*** (and where an arrow from f to g is a commutative square with f and g as the vertical sides). However another approach to ***Mul*** is to regard the *individual multisets* as categories, which indeed they are: if X is a multiset and $a, b \in X$, there is one arrow from a to b if a and b are of the same sort, and no arrows from a to b if a and b are of different sorts. In fact from the point of view of enriched category theory (see the book [3] by Kelly), ***Mul*** is the category of symmetric categories enriched over 2, where 2 is the two-element linearly ordered set regarded as a category, and a symmetric category X enriched over 2 is one such that $\hom(x, y) = \hom(y, x)$ for all $x, y \in X$. We will not go any further into enriched category theory here, but this shows that there is some point to regarding the individual multisets as categories.

The notions introduced in Definitions 2.10 and 2.11 can be treated from this point of view. The notion of replete multiset of Definition 2.10 corresponds to the notion of *replete full subcategory* of a category, where a replete full subcategory is one which, if it contains an object also contains all the isomorphs of that object. The operation $X \overset{+}{\cup} Y$ of Definition 2.11 can be explained as follows. Every functor $f\colon A \to B$ between arbitrary categories can be factored as $A \xrightarrow{g} C \xrightarrow{h} B$, where g is a functor which is a bijection on objects and h is fully faithful. When A and B are multisets, C is a multiset with the same elements as A, but a_1 and a_2 are of the same sort in C if and only if $f(a_1)$ and $f(a_2)$ are of the same sort in B. Now suppose that we have two multisets X and Y which are both submultisets of Z, so that we have strong monomorphisms $m_X\colon X \to Z$ and $m_Y\colon Y \to Z$. These combine to give a morphism $m\colon X \sqcup Y \to Z$. Factorize this morphism in the way just described into $X \sqcup Y \to C \to Z$. Then C is the multiset $X \overset{+}{\cup} Y$.

The fact that most of the multiset concepts considered in Section 2 generalize the corresponding concepts for ordinary sets can be considered from the following viewpoint. Call a multiset *discrete* if all its elements are of different sorts. The process which takes an ordinary set to the obvious discrete multiset is a fully faithful functor D: ***Set*** $\to$ ***Mul***. This functor D has both adjoints: the left adjoint takes a multiset to the set of sorts of that multiset, and the right adjoint U takes a multiset to its field. (U itself has a right adjoint which takes a set X to the *chaotic multiset* with X as field, where all elements are required to be of the same sort.) Since D has both adjoints, it preserves all limits and colimits. D also preserves exponentiation. D does not preserve the strong subobject classifier (since Ω is not discrete), but it does take $2 \in$ ***Set*** to the replete submultiset classifier $\dot{2}$.

How ***Set***-like is the category ***Mul***? One can approach this question in two ways. The first is to list properties which ***Mul*** has in common with ***Set***. Thus ***Mul*** is (small-) complete and cocomplete, and cartesian closed. However ***Mul*** is not a topos, as not every monomorphism is strong. Indeed, although ***Mul*** has a strong subobject classifier, ***Mul*** is not even a quasitopos, as in ***Mul*** strong partial morphisms are not in general represented (see WYLER [7]). From this point of view, then, ***Mul*** is not very like ***Set***. A second approach, however, involves considering not only multiset morphisms (seen as functors), but also the natural transformations between functors. It is easy to see that if $f: X \to Y$ and $g: X \to Y$ are multiset morphisms, then f and g are naturally isomorphic if and only if $f(x)$ and $g(x)$ are of the same sort for every $x \in X$. This puts an equivalence relation on the set of morphisms from X to Y. If we let ***Mul'*** be the category with the same objects as ***Mul***, but having equivalence classes of multiset morphisms as arrows, then ***Mul'*** $\cong$ ***Set***. Thus ***Mul*** can be seen as ***Set*** "expanded" by the equivalence relation; ***Mul*** indeed appears as a category of *multisets*.

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